Chapter 6

1. Introduction

In this chapter you will explore the determinant function and some of its properties. We'll work with the following MATLAB commands: rand(), round(), det(), diag(), and triu().

2. Elementary Row Operations and the Determinant [6.2]

In this section, we'll use a random 4x4 matrix A to explore remarkable patterns between the determinant of A and the determinants of A altered by each of the three EROs.

For convenience, we'll build a random 4x4 matrix row by row:

```
>> v1 = round(10*(2*rand(1,4)-1));
>> v2 = round(10*(2*rand(1,4)-1));
>> v3 = round(10*(2*rand(1,4)-1));
>> v4 = round(10*(2*rand(1,4)-1));
>>  A = [v1; v2; v3; v4]
```

MATLAB tricks: **rand(m,n)** creates an mxn matrix with random real entries between 0 and 1.
round(10*(2*rand(m,n) – 1) creates an mxn matrix with random integer entries between –10 and 10.

```
A =

   3    -6     7    -7
  -7    10    -1    -3
  -4    -3    -2     2
  -8    -9    -1     7
```

Note: The matrix A in your Command Window will likely be different from the matrix shown on the left.

The MATLAB command det(A) computes the determinant of the matrix A.

```
>> det(A)

ans =

        2640
```

Recall the three EROs:

1. Swap two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of one row to another.

ERO #1
Check out the determinant of the matrix made by swapping row one and row two:

```
>> det([v2; v1; v3; v4])
```

2a. Create two other matrices from A by swapping other pairs of rows and compute the determinant of each new matrix.

Conjecture 1
If matrix B is obtained from matrix A by performing one row swap, then det(B) = ______

2b. 6.1 #57

ERO #2
Check out the determinant of the matrix made by multiplying row three of A by 2:

```
>> det([v1; v2; 2*v3; v4])
```

2c. Define a new variable using >>sym, multiply any row of A by your variable, then compute the determinant of the new matrix.

Conjecture 2
If matrix B is obtained from matrix A by multiplying one row of A by a nonzero constant k, then det(B) = ______________________

ERO #3
Check out the determinant of the matrix made by replacing row four with –2 times row three plus row four:

```
>> det([v1; v2; v3; -2*v3 + v4])
```

2d. With the matrix A, add a nonzero multiple of one row to another and compute the determinant of the new matrix. (Try it also with the variable you used in problem 2b above.)

Conjecture 3
If matrix B is obtained from matrix A by adding a multiple of one row of A to another row of A, then det(B) = ________________

2e. 6.1 #43
2f. 6.1 #44

3. Summary and Comparison of det(A) with det(rref(A))

Suppose B is the matrix obtained by performing one ERO on A. Based on your conjectures in section 2, we have det(A) = c* det(B) where $c \neq 0$. Continuing to perform EROs on B yields det(A) = m*det(rref(A)) for some $m \neq 0$. From this we conclude that

$$\det(A) = 0 \Leftrightarrow \det(\mathrm{rref}(A)) = 0$$

3a. Suppose the square matrix A has a row of zeros. Explain why det(A) = 0.

3b. Suppose the square matrix A has linearly dependent rows. Explain why det(A) = 0.

4. What Does the Determinant Have to Do with Unique Solutions?

Recall that rref(A) is an upper triangular matrix. In this section we'll extend your conjectures in section 2 to a general understanding of the determinant applied to upper triangular matrices and ultimately reach a connection between det(A) and solutions to the linear system $A\vec{x} = \vec{b}$.

4a. Compute det(eye(n)) for n = 2, 3, 4, 5, and 6.

Conjecture 4
det(eye(n)) = _____________ for any integer n > 1.

4b. Based on conjectures 2 and 4, what would you predict for det(diagonal matrix)? Explain. To test your prediction, you can experiment with MATLAB's diag() command. diag(n-vector) returns an nxn diagonal matrix with n-vector's components along the main diagonal. It works with either row vectors or column vectors. For example,

```
>> r1 = round(10*(2*rand(1,4)-1))

r1 =

    -2     9    -9     1
```

```
>> diag(r1)

ans =

    -2     0     0     0
     0     9     0     0
     0     0    -9     0
     0     0     0     1
```

…or by columns…

```
>> c1 = round(10*(2*rand(3,1)-1))

c1 =

    -4
     7
    -3

>> diag(c1)

ans =

    -4     0     0
     0     7     0
     0     0    -3
```

Conjecture 5
det(diagonal matrix) = ____

4c. Based on conjectures 2, 3, and 4, what would you predict for det(upper triangular matrix)? Explain. (Hint: Imagine starting with an nxn identity matrix and use EROs to build an upper triangular matrix.) To test your prediction, experiment with MATLAB's triu() command. For example,

```
>> triu(round(10*(2*rand(3)-1)))

ans =

    -5     0     2
     0    -2    -9
     0     0    -4
```

Conjecture 6
det(upper triangular matrix) = ______________________________

4d. Based on conjecture 6, formulate the next conjecture…

Conjecture 7
det(upper triangular matrix with a zero on the main diagonal) = ____________

We now have enough ingredients to make the following chain of equivalencies:

det(A) = 0 ⇔ det(rref(A)) = 0

⇔ rref(A) has a zero on its main diagonal

⇔ rref(A) $\neq$ I

⇔ A is not invertible

⇔ the nxn linear system $A\vec{x} = \vec{b}$ does not have a unique solution

4e. State the contrapositive version of this list of equivalencies. For example,

det(A) $\neq$ 0 ⇔ det(rref(A)) $\neq$ 0

⇔ rref(A) has no zeros on its main diagonal

⇔ ______________________________

⇔ ______________________________

⇔ ______________________________

____

4f. 6.1 #30 (Hint: MATLAB's solve(polynomal) will return the roots of a polynomial. For example,

```
>> syms x
>> solve(x^2 - 1)

ans =

[  1]
[ -1]
```

5. Some Properties of the Determinant

In this section we'll examine how the determinant behaves with other alterations to a matrix, namely, transposes, inverses and products of matrices.

Create two random 6x6 matrices C and D:

```
>> C = round(10*(2*rand(6)-1));
>> D = round(10*(2*rand(6)-1));
```

5a. Compute the following:

det(C*D) = _______________ det(C)*det(D) = _________________

(Reminder: The rats() command will convert outputs to more normal looking numbers.)

Conjecture 8
For any two square matrices A and B, det(A*B) = ____________________

5b. With inv(C) representing the inverse matrix of C, use conjecture 8 to predict det(inv(C)). Explain. (Hint: What's C*inv(C)?)

Conjecture 9
For any invertible matrix A, det(inv(A)) = _________________________________

5c. Compute the following:

det(C) = ________________ det(C') = ________________

det(D) = ________________ det(D') = ________________

Conjecture 10
For any square matrix A, det(A') = ______________________________

5d. In light of conjecture 10, restate conjectures 1, 2, and 3 in terms of elementary *column* operations.

Conjecture 11

____

Conjecture 12

____

Conjecture 13

____

6. Geometrical Interpretations of the Determinant [6.3]

Review Definition 6.3.6 on page 278. Note that 1-volume means length, 2-volume means area, 3-volume means the usual volume and for $m > 3$, m-volume is what we call hypervolume.

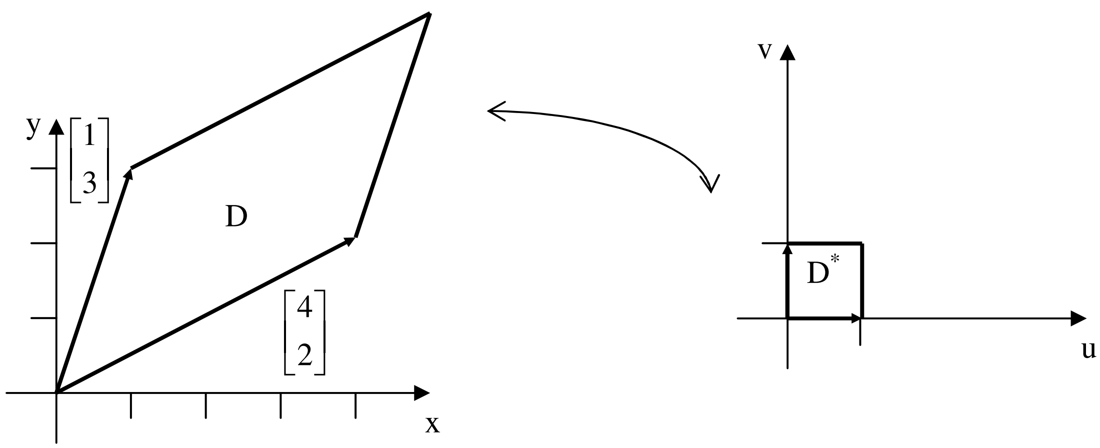
For example the hypervolume of the 4-parallelopiped defined by the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ in R^n ($n \geq 4$) is given by the recursively defined formula

$$\begin{aligned} V(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) &= V(\vec{v}_1, \vec{v}_2, \vec{v}_3) \| \vec{v}_4^{\perp} \| \\ &= V(\vec{v}_1, \vec{v}_2) \| \vec{v}_3^{\perp} \| \| \vec{v}_4^{\perp} \| \\ &= V(\vec{v}_1) \| \vec{v}_2^{\perp} \| \| \vec{v}_3^{\perp} \| \| \vec{v}_4^{\perp} \| \\ &= \| \vec{v}_1 \| \| \vec{v}_2^{\perp} \| \| \vec{v}_3^{\perp} \| \| \vec{v}_4^{\perp} \| \end{aligned}$$

6a. Suppose the three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in R^3 are linearly dependent. Let A be the 3x3 matrix with columns $\vec{v}_1, \vec{v}_2, \vec{v}_3$. From conjecture 10 and the result of problem 3b above one can show det(A) = 0. Give a geometric argument for why det(A) = 0. (Hint: See Fact 6.3.7 on page 279.)

6b. 6.3 #14

In multivariable calculus under the topic of integration and change of variables there is a need to keep track of how areas become altered by transformations. For example, say the region D in the x-y plane shown below is inconvenient from a calculus standpoint and the linear change of variables $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$ corresponds to a more convenient region D^* in the u-v plane. Notice you can easily build the transformation matrix (that actually transforms (*u, v*) coordinates to (*x, y*) coordinates) from the vectors that define the parallelogram D.



Correcting for the altered area in the new coordinate system, we have

$$\text{Area of D} = |\det(A)| \text{ Area of } D^*$$

$$\text{Area of D} = (10)\text{Area of } D^*$$