Chapter 4

1. Introduction

This chapter is about generalizing the concepts you learned in chapter 3 to spaces other than $\Re^n$. Many topics in this chapter are theoretical and MATLAB will not be able to help you out. You will see where MATLAB is useful in chapter 4 and how to use it in those places. You will also learn how to use MATLAB to differentiate and integrate.

There is one basis idea floating through most this lab: Many questions in Linear Algebra can be answered by solving a set of equations. MATLAB can be used to set up the equations and can then be used to solve them.

2. Linear Combinations [4.1]

In the previous chapters, you learned that given a set of vectors (usually a basis), you can build more vectors with linear combinations of the original vectors. The key idea to understand chapter 4 is that this idea can be applied to other types of mathematical objects. For example,

- linear combinations of 2x2 matrices are 2x2 matrices
- linear combinations of polynomials are polynomials
- linear combinations of continuous functions are continuous function

etc.

We can use MATLAB to determine the correct coefficients to use when building the linear combinations. Here is 4.1, example #9:

Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$. Show that $A^2 = \begin{bmatrix} 2 & 3 \\ 6 & 11 \end{bmatrix}$ is a linear combination of A and I_2.

As explained in the text, we need to find scalars c_1 and c_2 such that $\begin{bmatrix} 2 & 3 \\ 6 & 11 \end{bmatrix} = c_1 \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. To use MATLAB, it is easier to change the equations so that one side is 0. We will use MATLAB to solve

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = c_1 \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 6 & 11 \end{bmatrix}$$

```
>> syms c1 c2
>> c1*[0 1; 2 3] + c2*[1 0; 0 1] - [2 3; 6 11]
```

```
ans =

[        c2-2,        c1-3]
[     2*c1-6, 3*c1+c2-11]
```

This gives us the following equations:

$$\begin{array}{ccccccc} & & c_2 & - & 2 & = & 0 \\ c_1 & & & - & 3 & = & 0 \\ 2c_1 & & & - & 6 & = & 0 \\ 3c_1 & + & c_2 & - & 11 & = & 0 \end{array}$$

In your head, you can see that $c_1 = 3$ and $c_2 = 2$, so $\begin{bmatrix} 2 & 3 \\ 6 & 11 \end{bmatrix} = 3\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
If the system of equations was more complicated, you know from previous chapters how to use MATLAB to solve a system of equations.

Problems

2a. Express $\begin{bmatrix} 5 & 8 \\ 7 & 10 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$.
2b. Express $\begin{bmatrix} 8 & 2 \\ 2 & -8 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 7 & 8 \end{bmatrix}$.

3. Bases [4.1]

To find a basis for a given space, do the following:

1) Make a generic object
2) Write down the relation(s) the generic object must satisfy
3) Use MATLAB to determine a set of equations the parameters need to satisfy
4) Use MATLAB to solve the set of equations (if necessary)
5) Parameterize your solution and plug it back into the generic object.
6) Factor out the parameters to determine your basis elements.

Here is 4.1 example #17:

Find a basis of the space V of all matrices B that commute with $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$

Following your author, first we build a generic object. (1)In this case, the generic object is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then we write down the relation that it needs to satisfy.

(2) Here, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. To use MATLAB, it is easier to write this as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(3) Now use MATLAB to get the equations.

```
>> syms a b c d
>> [a b; c d]*[0 1; 2 3] - [0 1; 2 3]*[a b; c d]

ans =

[       2*b-c,      a+3*b-d]
[ 2*d-2*a-3*c,        c-2*b]
```

The equations are

$$\begin{array}{rcrcrcrcl} & & 2b & - & c & & & = & 0 \\ a & + & 3b & & & - & d & = & 0 \\ -2a & & & - & 3c & + & 2d & = & 0 \\ & & -2b & + & c & & & = & 0 \end{array}$$

(4) Use MATLAB to solve the system. Here we look at the row reduced echelon form of the augmented matrix.

```
>> rats(rref([0 2 -1 0; 1 3 0 -1; -2 0 -3 2; 0 -2 1 0]))

ans =

       1              0              3/2            -1
       0              1             -1/2             0
       0              0               0              0
       0              0               0              0
```

The nontrivial equations are

$$\begin{array}{rcrcrcrcl} a & & & + & \frac{3}{2}c & - & d & = & 0 \\ & & b & - & \frac{1}{2}c & & & = & 0 \end{array}$$

(5) The free variables are *c* and *d*, so we get the general solution $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}s+t \\ \frac{1}{2}s \\ s \\ t \end{bmatrix}$.

Plugging back into our generic object, we get

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}s+t & \frac{1}{2}s \\ s & t \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}s & \frac{1}{2}s \\ s & 0 \end{bmatrix} + \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} = s\begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} + t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(6) So a basis is $\left(\begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$. If you prefer your bases to not have fractions, you can multiply the first element by 2 to get $\left(\begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$ as your basis. Note that this is a different basis than the one obtained by your textbook.

Let's do one more. Here is 4.1 #3:

Find a basis for the subspace of P_2 where $\{p(t) : p'(1) = p(2)\}$.

(1) $p(x) = ax^2 + bx + c$

(2) $p'(1) = p(2)$. We will write this as $p'(1) - p(2) = 0$

(3)

MATLAB trick: MATLAB gives all of the variables in $ax^2 + bx + c$ the same weight. If you just told MATLAB to plug in a 2, it would not know whether the 2 was supposed to replace the *a*, *b*, *c*, or *x*. Use the `subs` command to plug a value for a variable into a function.

MATLAB trick: Use `diff` to find the derivative of a function

```
>> syms a b c x
>> p = a*x^2 + b*x + c;
>> p_prime = diff(p);
>> subs(p_prime,x,1) - subs(p,x,2) % This is p'(1)-p(2)

ans =

-2*a-b-c
```

(5) The equation is $-2a-b-c=0$. *b* and *c* are the free variables. The parameterized solution is $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}s-\frac{1}{2}t \\ s \\ t \end{bmatrix}$ and the generic object is

$$\left(-\tfrac{1}{2}s-\tfrac{1}{2}t\right)x^2+s\,x+t=\left(-\tfrac{1}{2}sx^2+sx\right)-\left(\tfrac{1}{2}tx^2+t\right)=s\left(-\tfrac{1}{2}x^2+x\right)+t\left(-\tfrac{1}{2}x^2+1\right)$$

(6) A basis is $\left(-\frac{1}{2}x^2+x,-\frac{1}{2}x^2+1\right)$. Multiplying each polynomial by -2 gives us the better looking basis $\left(x^2-2x,\ x^2-2\right)$

Problems:

3a. 4.1 #4

MATLAB trick: Use the `int` command to integrate.

(Hint:

```
>> syms a b c x
>> p = a*x^2 + b*x +c;
>>int(p,0,1)
)
```

3b. A magic square is an $n\ x\ n$ matrix where the *n* numbers along any row, column and diagonal add up to the same constant sum. Try `>> magic(3)` to see an example. Try `>> magic(3) + ones(3,3)` to see another.

First, convince yourself that the set of all 3 x 3 magic squares form a linear space. Now, find a basis for this space. (Hint: Your generic object will have 9 parameters.)

4. Isomorphisms [4.2]

An isomorphism is a linear transformation that shows that two linear spaces have the same structure. Given a linear transformation, it is important to determine whether or not it is an isomorphism. The top of pg. 169 your textbook has a nice flow chart to help you determine whether a given linear transformation is an isomorphism. Unfortunately, MATLAB can't do all of this for us, so we settle for just a part. We use the following fact.

> Given *T*, a linear transformation from *V* to *W*,
> *T* is an isomorphism iff dim(*V*) = dim(*W*) and ker(*T*) = $\{\vec{0}\}$.

Here is 4.2 #6

Determine whether $T(M) = M\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is an isomorphism where $T : \Re^{2x2} \to \Re^{2x2}$

First of all, notice that *V* and *W* have the same dimension (in fact, they are the same space). We need to find *ker*(*T*). In other words, we need to find all *M* such that

$$T(M) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

```
>> syms a b c d
>> M = [a b; c d];
>> M*[1 2; 3 6]

ans =

[   a+3*b, 2*a+6*b]
[   c+3*d, 2*c+6*d]
```

This means $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in $\ker(T)$ iff $\begin{bmatrix} a+3b & 2a+6b \\ c+3d & 2c+6d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. We have to see if

$$\begin{array}{ccccccccc} a & + & 3b & & & & & = & 0 \\ 2a & + & 6b & & & & & = & 0 \\ & & & & c & + & 3d & = & 0 \\ & & & & 2c & + & 6d & = & 0 \end{array}$$ has any solution besides $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Use MATLAB again,

```
>> rats(rref([1 3 0 0; 2 6 0 0; 0 0 1 3; 0 0 2 6]))

ans =

          1              3              0              0
          0              0              1              3
          0              0              0              0
          0              0              0              0
```

So, the nontrivial equations are

$$\begin{array}{ccccccccc} a & + & 3b & & & & & = & 0 \\ & & & & c & + & 3d & = & 0 \end{array}$$

Since *b* and *d* are the free variables we can parameterize the system by $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -3s \\ s \\ -3t \\ t \end{bmatrix}$.

By setting s = 0, *t* = 1, you can see that the system has nontrivial solutions. So, ker(*T*) $\neq \{\bar{0}\}$, and hence $T(M) = M\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is not an isomorphism.

Problems:

4a. 4.2 #7
4b. 4.2 #25

5. The Matrix of a Linear Transformation [4.3]

The key idea for section 4.3 is that every finite dimensional vector space is isomorphic to $\Re^n$ for some *n*. Moreover, given a basis for your finite dimensional vector space, then the isomorphism is extremely easy to build. Before we go on, let's do an example.

Suppose we are using the basis $\left(1, 1+x, 1+x+x^2\right)$ for P_2. Since P_2 is has dimension 3, P_2 is isomorphic to $\Re^3$. The isomorphism is the unique linear transformation where

$$1 \leftrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad 1+x \leftrightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad 1+x+x^2 \leftrightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Using this idea you can see that every linear transformation from a finite dimensional vector space to itself can be looked at as a linear transformation from $\Re^n$ to $\Re^n$. Because a linear transformation from $\Re^n$ to $\Re^n$ can be represented by a matrix, we can also use a matrix to represent any linear transformation from any finite dimensional vector space to itself.

Here is a neat example that will end with a way to calculate $\int e^x \sin(x)\, dx$ without using integration by parts. You don't really need MATLAB for this one, but it is worth going through.

Let V = span$(e^x \cos(x), e^x \sin(x))$. Let $T : V \to V$ be defined by $T(f) = f'$. We identify V with $\Re^2$ by the association

$$e^x\cos(x) \leftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad e^x\sin(x) \leftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We build up the matrix representing T column by column.

MATLAB trick: e^x is represented as `exp(x)`

```
>> syms x
>> diff(exp(x)*cos(x))

ans =

exp(x)*cos(x)-exp(x)*sin(x)
```

In other words, $T(e^x\cos(x)) = e^x\cos(x) - e^x\sin(x)$. Looking at this in terms of $\Re^2$,

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. This tells us that the first column of the matrix representing T is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

```
>> diff(exp(x)*sin(x))

ans =

exp(x)*sin(x)+exp(x)*cos(x)
```

In other words, $T(e^x\sin(x)) = e^x\cos(x) + e^x\sin(x)$. Looking at this in terms of $\Re^2$,
$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. This tells us that the second column of the matrix representing T is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Putting it all together, the matrix representing T is $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Since T is the differentiation function, T^{-1} integration. We use MATLAB to find the matrix that represents T^{-1}.

```
>> T = [1 1; -1 1];
>> rats(inv(T))

ans =

    1/2       -1/2
    1/2        1/2
```

Since $e^x \sin(x) \leftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we can find $\int e^x \sin(x)\,dx$ using the computation

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Interpreting, we see that $\int e^x \sin(x)\,dx = -\frac{1}{2} e^x \cos(x) + \frac{1}{2} e^x \sin(x) + C$.

Here's one more example:

> Find a matrix to represent the linear transformation $T(f) = f + f' + f''$ from P_2 to P_2 with respect to the basis $(1, x, x^2)$

We make the association

$$1 \leftrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad x \leftrightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad x^2 \leftrightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and compute the matrix one column at a time.

MATLAB trick: MATLAB does not differentiate constant functions in the obvious way. Note the syntax below.

```
>> syms x
>> b1 = 1;
>> b1 + diff(b1,x) + diff(diff(b1,x),x)  % This is the transformation
T applied to the constant function f(x) = 1

ans =

1
```

Since $T(1) = 1 \leftrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, the first column of the matrix representing T is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

```
>> b2 = x;
```

```
>> b2 + diff(b2) + diff(diff(b2)) %MATLAB knows what to do
when the function has the variable x.

ans =

x+1
```

Since $T(x) = 1 + x \leftrightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, the second column of the matrix representing T is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

```
>> b3 = x^2;
>> b3 + diff(b3) + diff(diff(b3))

ans =

x^2+2*x+2
```

Since $T(x^2) = 2 + 2x + x^2 \leftrightarrow \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, the third column of the matrix representing T is $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

Thus, the matrix representing T with respect to the basis $(1, x, x^2)$ is $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

What if we wanted the matrix representing T with respect to a more exotic basis like $(1, 1 + x, 1 + x + x^2)$? You could do it by hand like we did above, or you could do something to the answer you obtained for the basis $(1, x, x^2)$. This second approach will be explored in the next section.

Problems

5a. 4.3 #34
5b. 4.3 #49

6. Change of Basis [4.3]

The last section ended with an answer to a question about the basis $(1, x, x^2)$. Now we want to use your matrix to answer the same question with respect to the basis $(1, 1 + x, 1 + x + x^2)$. Luckily, we can use a matrix to transform from one basis to another.

Then we can combine the change of basis matrix and the matrix found in the last section with matrix multiplication.

First, we build the change of basis matrix for changing from basis B = $(1, 1+x, 1+x+x^2)$ to basis S = $(1, x, x^2)$.

As usual, we do it column by column.

The first entry in B is 1. The coordinate matrix of 1 relative to S is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so the first column of the change of basis matrix is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

The second entry in B is $1+x$. The coordinate matrix for $1+x$ relative to S is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, so the second column of the change of basis matrix is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

The third entry in B is $1+x+x^2$, so the third column of the change of basis matrix is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

The change of basis matrix from B to S is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. The change of basis matrix in the other direction (from S to B) is the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Now we'll build the matrix for the transformation relative to $(1, x, x^2)$. Before we go on, let's review what we have so far:

Relative to the basis $(1, x, x^2)$, the matrix representing the transformation $T(f) = f + f' + f''$ is $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Call this matrix A.

The change of basis matrix from the basis $(1, 1+x, 1+x+x^2)$ to the basis $(1, x, x^2)$ is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Call this matrix S.

The key idea for using these matrices to compute the matrix that represents the transformation $T(f) = f + f' + f''$ relative to the basis $(1, 1+x, 1+x+x^2)$ is to think about how you would actually do the computation yourself given the coordinates of an element in P_2 relative to $(1, 1+x, 1+x+x^2)$:

1) Use S to change to the basis $(1, x, x^2)$.
2) Use A to compute T.
3) Use S^{-1} to change your answer back to coordinates relative to $(1, 1+x, 1+x+x^2)$.

In symbols, given $\begin{bmatrix} a \\ b \\ c \end{bmatrix}_B$,

1) $S\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

2) $AS\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

3) $S^{-1}AS\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

So the matrix that does the job is $S^{-1}AS$.

```
>> A = [1 1 2; 0 1 2; 0 0 1];
```

```
>> S = [1 1 1; 0 1 1; 0 0 1];
>> inv(S)*A*S

ans =

     1     1     1
     0     1     2
     0     0     1
```

Wrapping it up: The matrix representing the transformation $T(f) = f + f' + f''$ from P_2 to P_2 with respect to the basis $\left(1, 1+x, 1+x+x^2\right)$ is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

Problems

6a. 4.3 #14
6b. 4.3 #24