Chapter 3

1. Introduction

In this lab we explore the ideas in chapter 3. We'll use the following MATLAB commands as you progress through this section: `rref, null, rats, inv`

2. The Image and Kernel of a Linear Transformation [3.1]

Thinking of matrix-vector multiplication as, for example,

$$A\vec{x} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

we see that the "outputs" of multiplication by A are combinations of the column vectors of A. But since Col.3 is 2*Col.2 – Col.1, we see that Col.3 is redundant, so the image of A could be spanned by the 1st, 2nd, and 4th columns of A. Later on we'll use MATLAB to find a "minimal spanning set" of column vectors. It's common in linear algebra to describe a space using a representative set of vectors.

Next let's use MATLAB to find the kernel of the matrix A above. Keeping in mind that the kernel of a linear transformation (or matrix) is the stuff in the domain that gets sent to $\vec{0}$, determining the kernel is equivalent to solving the matrix equation $A\vec{x} = \vec{0}$. We can use the `rref` command in MATLAB on just the coefficient matrix rather than setting up the usual augmented matrix. The last column of the augmented matrix would be all zeros, and so remains unchanged under the elementary row operations.

We get,

```
>> rref(A)

ans =

     1     0    -1     0
     0     1     2     0
     0     0     0     1
```

which corresponds to the system

$$\begin{array}{rcrcrcl} x_1 & & - & x_3 & & = & 0 \\ & x_2 & + & 2x_3 & & = & 0 \\ & & & & x_4 & = & 0 \end{array}$$

This indicates that $x_4 = 0, \quad x_3 = t, \quad x_2 = -2t, \quad x_1 = t$

The kernel could then be described by $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$. We have two equivalent ways to phrase this:

1) The kernel is all linear combinations[1] of $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$.

2) The kernel of A is spanned by $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$.

Geometric observation: Note the kernel here is a line in R^4 that passes through the origin and the point (1, -2, 1, 0).

Use MATLAB to help answer the following problems:

2a. 3.1 #37.

2b. 3.1 #41.

2c. 3.1 #42. Use MATLAB to compute the rref of the "super augmented" matrix and verify that the rref given by MATLAB matches the form given at the bottom of page 110. You can build the super augmented matrix in several ways. Other than entering the matrix directly with the first row as 1 1 1 6 1 0 0 0, etc., you can type the following:

```
>> A = [1 1 1 6; 1 2 3 4; 1 3 5 2; 1 4 7 0];
>> I = eye(4);
>> superAugmented = [A I]
```

MATLAB trick: `eye(n)` creates an nxn *identity matrix.*

3. Subspaces of R^n, Bases and Linear Independence [3.2]

Reminders:

- A set of vectors qualifies as a *subspace* of R^n iff the collection is closed under scalar multiplication and closed under addition.

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[1] Of course, for a single vector this means "all scalar multiples."

- A set of representative vectors works as a *basis* for a subspace of R^n, if the set is linearly independent and spans the subspace.

By the way we found the kernel of the matrix in section 2, and with Fact 3.2.5 in the text, the representative vector[2] we discovered actually serves as a **basis for the kernel** of the transformation (or matrix).

Example: Find a basis for the kernel of the matrix $V = \begin{bmatrix} 1 & 2 & 5 & -2 & 1 \\ -1 & 2 & 3 & 2 & 3 \\ 4 & -1 & 2 & -1 & 2 \\ 2 & -1 & 0 & 3 & 4 \end{bmatrix}$

The MATLAB output for the *rref*(*V*) is shown below

```
>> rref(V)

ans =

     1     0     1     0     1
     0     1     2     0     1
     0     0     0     1     1
     0     0     0     0     0
```

which corresponds to the system
$$\begin{array}{ccccccccccc} x_1 & & & + & x_3 & & & + & x_5 & = & 0 \\ & & x_2 & + & 2x_3 & & & + & x_5 & = & 0 \\ & & & & & & x_4 & + & x_5 & = & 0 \\ & & & & & & & & 0 & = & 0 \end{array}$$

We have $x_5 = t, \quad x_4 = -t, \quad x_3 = s, \quad x_2 = -t - 2s, \quad x_1 = -t - s$, which we write as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -t-s \\ -t-2s \\ s \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ 0 \\ -t \\ t \end{bmatrix} + \begin{bmatrix} -s \\ -2s \\ s \\ 0 \\ 0 \end{bmatrix} = t\begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + s\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

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[2] The representative vector we used was the one left over after factoring out the parameter.

Since the vectors $\begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ are independent and span the kernel, we can choose them for a basis for the kernel of *V*.

3a. Find a basis for the kernel of the matrix $A = \begin{bmatrix} 1 & 0 & 3 & 5 & 6 \\ 1 & 1 & 2 & 0 & 4 \\ 3 & 2 & 7 & 5 & 14 \\ 2 & 1 & 5 & 5 & 10 \\ 5 & 3 & 12 & 10 & 24 \end{bmatrix}$

Determining a **basis for the image** of a transformation (or matrix) requires finding the smallest set of independent column vectors that still span the image. Below we describe a simple way to determine a basis for the image.

If you think of a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ as being independent if and only if the sole solution to $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v} = \vec{0}$ is the trivial solution $c_1 = c_2 = \cdots = c_n = 0$, then $rref\left(\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}\right)$ would have leading ones in all columns.

> Question: What must be true about the number of columns versus the number of rows to guarantee leading ones in all columns?

Recalling that the image of a matrix is all linear combinations of its columns, we'll work our way from left to right across the columns, building up larger matrices and checking the rref as we go. As redundant vectors are discovered, we eliminate those vectors from the set that will become a basis for the image.

For example, determine a basis for the image of $V = \begin{bmatrix} 1 & 2 & 5 & -2 & 1 \\ -1 & 2 & 3 & 2 & 3 \\ 4 & -1 & 2 & -1 & 2 \\ 2 & -1 & 0 & 3 & 4 \end{bmatrix}$

You can visually check whether the first two columns are independent (- they are, why?), so the first matrix we'll examine is made from the first three columns. For this type of work, it can be helpful to have the data organized as shown below.

```
>> v1 = [1 -1 4 2]';
>> v2 = [2 2 -1 -1]';
>> v3 = [5 3 2 0]';
>> v4 = [-2 2 -1 3]';
>> v5 = [1 3 2 4]';
>> rref([v1 v2 v3])

ans =

     1     0     1
     0     1     2
     0     0     0
     0     0     0
```

MATLAB trick: The apostrophe after the right bracket tells MATLAB to think of this row as a column. " ' " is called the *transpose operator*. In this case it's a little cleaner than typing `v1 = [1; -1; 4; 2];` etc.

So Col. 3 is a linear combination of the first two columns, that is, Col.3 is redundant. Eliminate Col.3 and build the next matrix from columns 1, 2, and 4.

```
>> rref([v1 v2 v4])

ans =

     1     0     0
     0     1     0
     0     0     1
     0     0     0
```

So columns 1, 2, and 4 are independent. Add Col. 5 to the matrix…

```
>> rref([v1 v2 v4 v5])

ans =

     1     0     0     1
     0     1     0     1
     0     0     1     1
     0     0     0     0
```

Therefore, Col. 5 is also redundant. So a basis for the image of our matrix could be made from the 1st, 2nd, and 4th columns of the matrix *V*.

This demonstration can be used to help justify Algorithm 3.3.5 described in the text on page 130.

3b. 3.2 #32
3c. 3.2 #46

4. The Dimension of a Subspace of $\mathbf{R}^n$ [3.3]

Defining the dimension of a subspace by saying it's the number of vectors in a basis relies on the idea that any basis for a space contains the same number of vectors. In this section we'll discover how you can come up with different bases for subspaces.

4a. Create a different basis for the image of $V = \begin{bmatrix} 1 & 2 & 5 & -2 & 1 \\ -1 & 2 & 3 & 2 & 3 \\ 4 & -1 & 2 & -1 & 2 \\ 2 & -1 & 0 & 3 & 4 \end{bmatrix}$ by cycling the columns around, that is, use `[v2 v3 v4 v5 v1]` and proceed as above. Cycle through a couple of more times. How many independent vectors are you finding each time?

4b. In section 3 above we found $\left(\begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right)$ as a basis for the kernel of the matrix $V = \begin{bmatrix} 1 & 2 & 5 & -2 & 1 \\ -1 & 2 & 3 & 2 & 3 \\ 4 & -1 & 2 & -1 & 2 \\ 2 & -1 & 0 & 3 & 4 \end{bmatrix}$. Use the `null(V)` command in MATLAB to discover another basis for the kernel of *V*. Check that MATLAB is indeed producing two independent vectors in the kernel of *V*.

5. Coordinates [3.4]

Now that you're thinking of coordinates as coefficients of basis vectors, consider the subspace H generated from the basis $\left(\vec{h}_1, \vec{h}_2\right) = \left(\begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \\ 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 1 \\ 8 \\ 2 \\ 8 \end{bmatrix} \right)$.

5a. What is the dimension of H? H is a subspace of what more general space? (That is, identify the dimension n in $\mathrm{H} \subseteq \mathrm{R}^n$.)

Reminder: In the context of bases, it is crucial to keep track of (and not change!) the order of the basis vectors. This is why we use parentheses around the sets of vectors that define a basis; because of the importance of order in a basis, one should think of a basis as *k*-tuples where each slot is occupied by a vector. For example, with respect to the basis $\Omega = \left(\vec{h}_1, \vec{h}_2\right)$, $\vec{h}_1$ would have coordinates $\begin{bmatrix} 1 \\ 0 \end{bmatrix}_\Omega$. And, with respect to Ω, the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}_\Omega$ would have standard coordinates representation $2\vec{h}_1 + 3\vec{h}_2 = \begin{bmatrix} 12 \\ 23 \\ 11 \\ 26 \\ 16 \\ 42 \end{bmatrix}$. In other words, if you're at the origin in R^6 but restricted to walking on the plane H, you can get to the point (12, 23, 11, 26, 16, 42) by walking in the direction $\vec{h}_1$ for twice its length, then turn so you're facing in the direction of $\vec{h}_2$ and walk three times its length.

5b. 3.4 #44.

5c. Determine whether the vector with standard representation $\vec{b} = \begin{bmatrix} 8 \\ -29 \\ 19 \\ -34 \\ 20 \\ 14 \end{bmatrix}$ is in the plane H, and if so, determine the coordinates of $\vec{b}$ with respect to Ω. (Hint: This problem is equivalent to solving the system $x_1\vec{h}_1 + x_2\vec{h}_2 = \vec{b}$.)

A common problem is to find the basis that will correspond to a desired change of coordinates. For example, suppose we want to find the basis $\Phi = \left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} f_3 \\ f_4 \end{bmatrix} \right)$ of R^2 such that $\begin{bmatrix} 1 \\ 3 \end{bmatrix}_\Phi = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}_\Phi = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. This can be viewed as a system of four unknowns

$$\begin{matrix} 1\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + 3\begin{bmatrix} f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ 3\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + 1\begin{bmatrix} f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \end{matrix}, \text{ or } \quad \begin{array}{cccccccc} f_1 & & + & 3f_3 & & & = & 2 \\ & f_2 & & & + & 3f_4 & = & 1 \\ 3f_1 & & + & f_3 & & & = & 3 \\ & 3f_2 & & & + & f_4 & = & 4 \end{array}$$

$$\text{or} \qquad \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \\ 3 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}.$$

Below is one way to use MATLAB to express the solution to this system using the rats() command that shows fractional answers:

```
>> c = [1 0 3 0; 0 1 0 3; 3 0 1 0; 0 3 0 1];
>> b = [2;1;3;4];
>> rats(rref([c b]))

ans =

     1         0         0         0        7/8
     0         1         0         0       11/8
     0         0         1         0        3/8
     0         0         0         1       -1/8
```

So $\Phi = \left(\begin{bmatrix} \frac{7}{8} \\ \frac{11}{8} \end{bmatrix}, \begin{bmatrix} \frac{3}{8} \\ \frac{-1}{8} \end{bmatrix} \right)$

5d. 3.4 #56.

Fact 2.1.2 on page 48 gave us a way to determine the matrix for a transformation *with respect to the standard basis*. At that point it did not need to be emphasized that we were finding the standard matrix for a transformation, since we only had the standard basis (we didn't even formally know about bases then!) Fact 3.4.4 on page 144 gives us a convenient way to determine the matrix of a transformation $T : R^n \to R^n$ with respect to any basis $(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$, if we are given the standard matrix for the transformation. Let A be the standard matrix for the transformation and B be the matrix for the transformation with respect to the new basis $(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$.

We summarize the argument for Fact 3.4.4 with the following diagram. In the diagram moving from left to right depicts the transformation and moving from top to bottom depicts the change of basis from standard to new representation. Let $S = [\vec{v}_1 \mid \vec{v}_2 \mid \cdots \mid \vec{v}_n]$.

$$\begin{tikzcd}
\vec{x} \arrow[r, "\text{Using the standard representation}"'] \arrow[d] & A\vec{x} \\
S^{-1}\vec{x} \arrow[r, "\text{Using the new basis}"] & BS^{-1}\vec{x} \arrow[u, "SBS^{-1}\vec{x} = A\vec{x}"']
\end{tikzcd}$$

To help see why we must apply S^{-1} to change coordinates from the standard basis to the new basis, consider the example below.

Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with respect to the standard basis and suppose we want to switch to a coordinate system that has basis $\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$. To express $\vec{x}$ with respect to the new coordinate system we need c_1 and c_2 such that

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Note that *applying S actually brings us from the new basis to the standard basis.* Therefore $SBS^{-1} = A$. Then to get the matrix version of the transformation with respect to the new basis, given the standard matrix and the new basis, we solve for B and get $B = S^{-1}AS$.

For example, let's use MATLAB to find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $(\vec{v}_1, \vec{v}_{2,}\ \vec{v}_3)$, where $\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ and $A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$. (This is the standard matrix of T.)

The MATLAB output is below.

```
>> A = [5 -4 -2;-4 5 -2;-2 -2 8];
>> v1 = [2;2;1];
>> v2 = [1;-1;0];
>> v3 = [0;1;-2];
>> S = [v1 v2 v3];
>> rats(inv(S)*A*S)

ans =

     0          0          0
     0          9          0      (This is the matrix B.)
     0          0          9
```

Note: We say A *is similar to* B whenever there is a matrix S such that S^{-1} AS = B.

5e. 3.4 #29.