# Chapter 2

## 1. Introduction

In this lab you will explore the ideas in chapter 2. You will do this from two perspectives: Coding and decoding a ship's position and the geometry of linear transformations.

## 2. The Product $\vec{y} = A\vec{x}$ [1.3/2.1]

Recall your adventures in the French coast guard from the text. If you remember, you had to code you position and transmit it to headquarters in Marseille. More specifically, suppose that your actual position is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and your encoded position is $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. For this problem we will assume $\begin{matrix} y_1 = 3x_1 + 2x_2 \\ y_2 = 4x_1 + 3x_2 \end{matrix}$. (1)

Another way to write this is $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. (2)

A third way to write this is $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. (3)

**Problems:**

You are going to verify that MATLAB's definition of matrix multiplication agrees with ours.

Suppose that your position is $\begin{bmatrix} 5 \\ 42 \end{bmatrix}$.

**2a.** Use definition (1) above and MATLAB to determine the code we send to headquarters.

```
>> 3*5 + 2*42
>> 4*5 + 3*42
```

What is the encoded position?

**2b.** Use definition (2) above to determine the code you send to headquarters.

```
>> 5*[3;4] + 42*[2;3]
```

What is the encoded position using this definition?

**2c.** Use definition (3) to determine the code.
(>> [3 2; 4 3]*[5; 42])

What is the encoded position using this definition?

**2d.** If your coding matrix is $\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$ and your actual position is $\begin{bmatrix} 10 \\ 25 \end{bmatrix}$, what is your encoded position?

## 3. Inverses. [2.1]

Now imagine the job of the person who receives your coded positions. He needs to determine your original position from your encoded position. More specifically, suppose your coding matrix is $\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$ and he receives the code $\begin{bmatrix} 80 \\ 350 \end{bmatrix}$. To find your actual position, the person receiving your code needs to solve the equations $\begin{aligned} 80 &= x_1 + 2x_2 \\ 350 &= 4x_1 + 9x_2 \end{aligned}$

**Problems:**

**3a.** Solve the equations to find the actual position.

**3b.** As described in the text (pgs. 42-43), decoding can also be done with a matrix. (You will learn more about this later in Chapter 2.) Read through the example and find the matrix that will do the decoding for you. For the record, the matrix that you find is called the inverse of $\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$.

**3c.** Some matrices are not good for coding. One of the matrices below is good, and one is bad. Which is the bad one? Why? (Hint: Read pgs 43-44 in the text).

$\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$

## 4. The Geometry of Linear Transformations [2.1]

In this section, you are going to work with the standard letter L from Example 2.1.5. The standard letter L is made up of the vectors $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. To draw the graph in MATLAB we need to put the corners of L into a matrix. The corners are $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and we use the matrix $\begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}$ to represent the L in MATLAB.

Here is example 5 on page 56. To see the graph of L, type the following into MATLAB:

```
>> L = [0 0 1; 2 0 0];
>> plot(L(1,:), L(2,:), 'LineWidth',5)
>> axis([-3 3 -3 3])
>> axis('square')
```

A graph of the letter L should appear in a new window. You might need to open the window to see it.

Now we apply the transformation $C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

```
>> C = [0 -1; 1 0];
>> newL = C*L;
>> plot(newL(1,:), newL(2,:), 'LineWidth',5)
>> axis([-3 3 -3 3])
>> axis('square')
```

You should now see the picture that is on the right side of example 5.

**Problems:**

**4a.** Pick a letter in your name other than the letter L.

i) Use MATLAB to draw your letter.

More about how MATLAB graphs:

MATLAB graphs $\begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}$ by drawing a line from $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and then drawing a line from $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Before you start drawing your letter, use MATLAB to graph $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$ to see how to make a closed curve.

ii) Apply the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ to your letter.

Include both pictures in your write-up.

**4b.** 2.1 #22. Use the matrix on the letter from your name instead of the standard letter L.

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| MATLAB trick: To save a copy of your graph, go to the menu at the top of the graph.<br>File → Export, and then save as type: Bitmap files. |

## 5. Linear Transformations in General [2.1]

The big idea in section 2.1, which you will see throughout this course, is that a matrix can be thought of as a function: the inputs are vectors and so are the outputs. For the coding, the inputs were the actual positions and the outputs were the encoded positions. For the letter L, the inputs were the two vectors that made up the legs, and the outputs were the legs in their shifted positions.

There is an easy way to build a matrix to represent any linear transformation you want. Just use images of the standard vectors $\vec{e}_1, \vec{e}_2, ..., \vec{e}_n$ to build the columns of your matrix. Here is an example.

Find the transformation that represents a rotation of $90^\circ$ clockwise.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, so the first column of the matrix is $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so the second column of the matrix is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

The matrix is thus $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

**5a.** Use the letter in your name from section 4 to verify that this matrix does what I claimed it did.

## 6. More Geometry of Linear Transformations [2.2]

In this section you will see how to represent special types of linear transformations as matrices. The thing to keep in mind for all of these examples is that you can build the matrix column by column by watching what the transformation does on the standard basis vectors.

### 6.1 Scaling

A scaling changes the size of a figure. Here is a scaling by a factor of 2 (all of the lengths will be doubled).

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, so the first column of the matrix is $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, so the first column of the matrix is $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

The matrix is thus $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

**6a.** Verify that this matrix does what is claimed on the standard letter L.

**6.2 Projections**

Given a line $L$ through the origin, we can project any vector $\bar{x}$ onto it.

**6b.** Read through pgs 57-58 in the text.

i) What matrix would project a vector onto the line $y = 2x$ ?
(Hint: You will need to find a unit vector on the line. A vector is on the line iff its $2^{nd}$ coordinate is twice its first coordinate.)

ii) Use your matrix and MATLAB to generate a picture of the standard letter L onto the line $y = 2x.$ by multiplying by a matrix.

**6.3 Reflections**

Given a line $L$ through the origin, we can also reflect any vector around it.

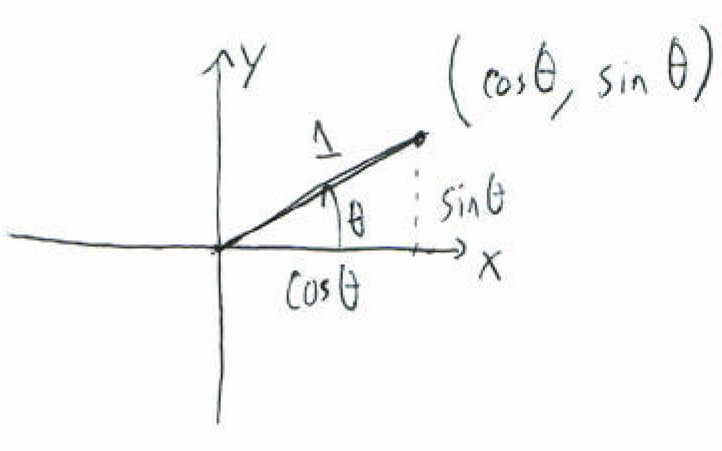
**6c.**
i) What matrix would reflect a vector about the line $y = 2x$ ?

ii) Use MATLAB and your answer above to find the coordinates for the reflection of the vector $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ about the line $y = 2x$ .

**6.4 Rotations**

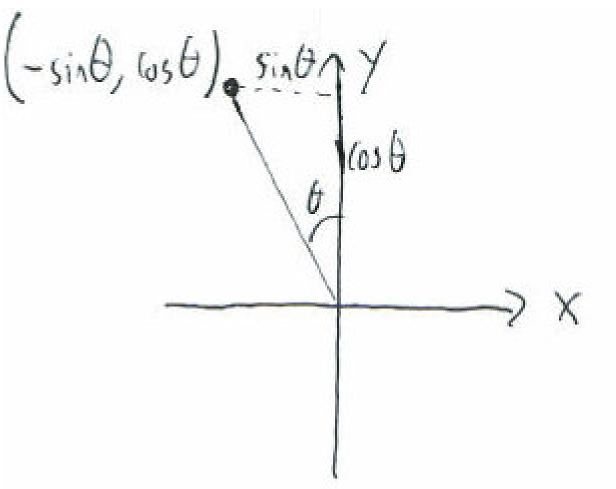
The last type of transformation we will consider in this section is rotation through any angle. We will develop an alternate way of determining the matrix.

First, we rotate $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ through a fixed angle $\theta$.




$$\begin{bmatrix}1\\0\end{bmatrix} \mapsto \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}$$

Then we rotate $\begin{bmatrix}0\\1\end{bmatrix}$ through a fixed angle $\theta$.




$$\begin{bmatrix}0\\1\end{bmatrix} \mapsto \begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\end{bmatrix}$$

So the rotation matrix is $\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}$

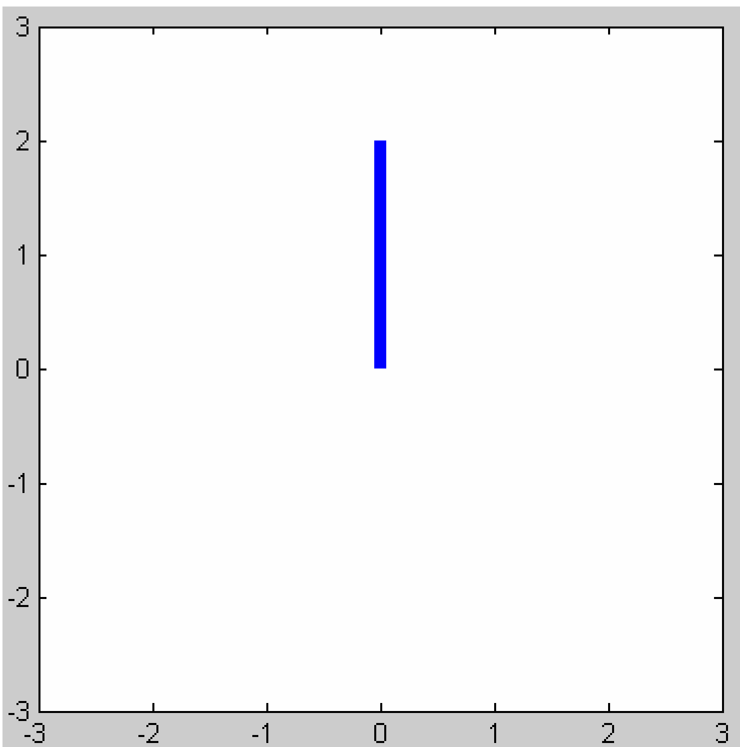
**6d.** Rotate the standard letter L counterclockwise by $\frac{\pi}{3}$.

(Hints: `>>help cos` MATLAB uses radians, not degrees . You can use expressions as part of your matrix. To build the matrix $\begin{bmatrix}\cos(2\pi) & 0\\ 1 & 1\end{bmatrix}$ you would type

```
>> [cos(2*pi) 0; 1 1] )
```

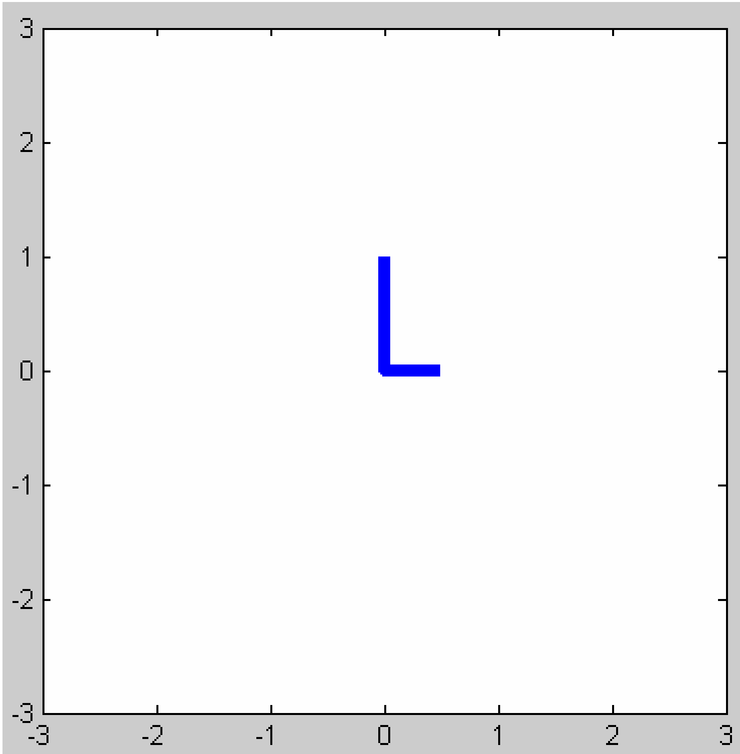
For the rest of the problems in this section, you will be given the result of applying a linear transformation to the standard letter L and you will need to find the matrix to represent the transformation.

**6e. I** applied the transformation *A* on the standard letter L and got the following picture.



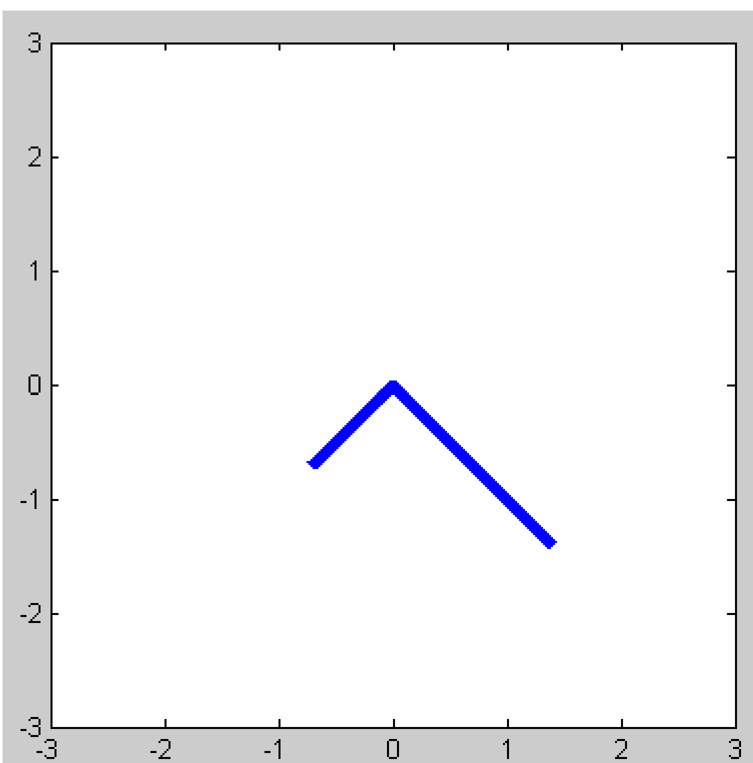

What is *A*?

**6f.** I applied the transformation *B* on the standard letter L and got the following picture:




Note that the lengths of each of the legs have changed. What is *B*?

**6g.** I applied the transformation *D* on the standard letter L and got the following picture:




What is *D*?

## 7. Computing Inverses [2.3]

In this section, you will use MATLAB to help you compute inverses. The first two problems will show you what is going on "under the hood" with the matrix computations and the third will show you how to use MATLAB to painlessly find inverses.

Recall that a square matrix is invertible iff its reduced row echelon form is the identity matrix.

**7a.** Use the MATLAB command `rref` to determine whether $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is invertible.

(Hint: `>> help rref` to see the fast way to get a matrix into reduced row echelon form.)

**7b.** Use the MATLAB command `rref` to find the inverse of $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

(Hint #1: See fact 2.3.5 on page 75)
(Hint #2:
```
>> A = [1 2 2; 1 3 1; 1 1 2];
>> I = [1 0 0; 0 1 0; 0 0 1];
>> B = [A I]
```

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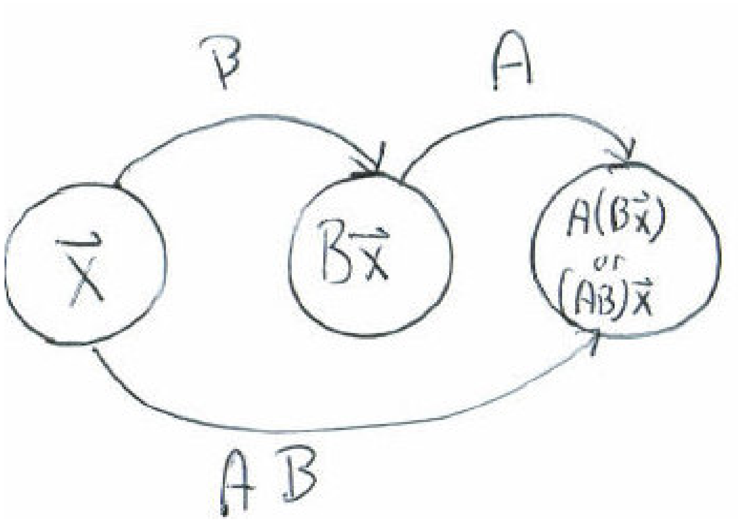
| MATLAB trick: An easier way to enter the 3x3 identity matrix is `>> eye(3)`. |

**7c**. 2.3 #38 (Hint: Type `>> help inv)`. You may want to review `sym` from the Chapter 1 lab.

## 8. Matrix Products and Inverses [2.4]

In this section we verify that MATLAB's definition of matrix multiplication is correct.

Matrix multiplication is defined so that $A(B\vec{x}) = (AB)\vec{x}$ for all $\vec{x}$. Here's a picture to make that clearer:




Following the top path: We start with $\vec{x}$, apply $B$ to $\vec{x}$ to get $B\vec{x}$, and then apply $A$ to $B\vec{x}$ and get $A(B\vec{x})$

Following the bottom path: We start with $\vec{x}$ and apply the matrix $AB$ to it. We define $AB$ so that we always end up in the same place.

**8a.** Let A = $\begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix}$, let B = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and let $\vec{x} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$.

i) Use MATLAB to compute $B\vec{x}$. What do you get?

ii) Use your answer from i) and MATLAB to compute $A(B\vec{x})$ What do you get?

iii) Let $C = AB$. (`>> C = A*B;`) Use MATLAB to compute $C\vec{x}$. How does your answer compare with $A(B\vec{x})$ that you computed in ii) above?

**8b.** Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and let A be Let A be as in 8a above. Use MATLAB to compute $AI$ and $IA$. What do you notice? Explain why this is reasonable.

**8c.** Let B be as in 8a above. Use MATLAB to compute $B^{-1}B$ and $BB^{-1}$. What do you notice? Explain in terms of inverses and composition of functions.

**8d.** Pick the letter in your name other than the letter L that you worked with in 4a above.

i) Use MATLAB to draw your letter.

ii) Find a single matrix that will reflect your letter about the line $y = -x$ , rotate it by an angle of $\frac{\pi}{2}$ clockwise, and then reflect about the line $y = 2x$ .

iii) Apply the matrix to the letter you chose above?

**8e:** Working for the French coast guard, you find that the smugglers are also using a matrix to encode their position. You find that when their position is $\begin{bmatrix} 10 \\ 120 \end{bmatrix}$ they send the code $\begin{bmatrix} 140 \\ 510 \end{bmatrix}$ and when their position is $\begin{bmatrix} 5 \\ 125 \end{bmatrix}$ they send the code $\begin{bmatrix} 135 \\ 515 \end{bmatrix}$.

i) What is their coding matrix? Show your work.

ii) What is the decoding matrix? Show your work.